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A Study of Statistical Errors in MICE

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Abstract

The Muon Ionization Cooling Experiment (MICE) will measure ionization cooling from a beam of muons at the Rutherford Appleton Laboratory in the UK. The aim of MICE is to measure a fractional drop in emittance, due to ionization cooling, of order 10% for a range of emittances and momenta, to an accuracy of 1%. A greater understanding of the statistical (as well as systematic) errors on emittance measurement in MICE is paramount to meeting this goal.

This paper describes a study aimed at exploiting the computing power of the Grid to determine the number of muons necessary to meet the scientific goals of MICE. In this study, tens of thousands of G4MICE Monte Carlo simulations were run to determine the scaling laws that govern the fractional change in emittance as a function of the number of muons (N) in the simulation. By varying random conditions, the standard deviation of these distributions was studied as a function of N. The results of the study indicate that, due to the effect of correlations, of order 10^5 muons are required to meet the goal of MICE for large emittance beams, without which 10^6 would be required.

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1 Introduction

The Muon Ionization Cooling Experiment (MICE) [1, 2], shown in Figure 1, aims to experimentally demonstrate ionization cooling: the reduction in four dimensional transverse normalised emittance[1)] of a muon beam due to ionization through a low Z absorber such as liquid hydrogen and subsequent re-acceleration via radio-frequency (RF) cavities. MICE is an experiment being built at the Rutherford Appleton Laboratory (RAL) in the UK and aims to demonstrate a fractional muon emittance reduction of 10% with an error of 1% (absolute emittance precision of 0.1%) for a variety of muon beams between 2π mm rad and 10π mm rad emittance and with momenta between 140 and 240 MeV/c. The muons cross three low density absorbers (containing liquid hydrogen or lithium hydride) inside a focusing magnetic field, where they experience a loss of total energy through ionization, and the longitudinal component of their momenta is restored through eight RF accelerating cavities inside two coupling coils. MICE measures the emittance before and after the cooling channel with two scintillating fibre trackers inside solenoidal magnetic fields, and particle identification detectors which are used to ensure that the particles contributing to the measurement are muons. For more details see [3].

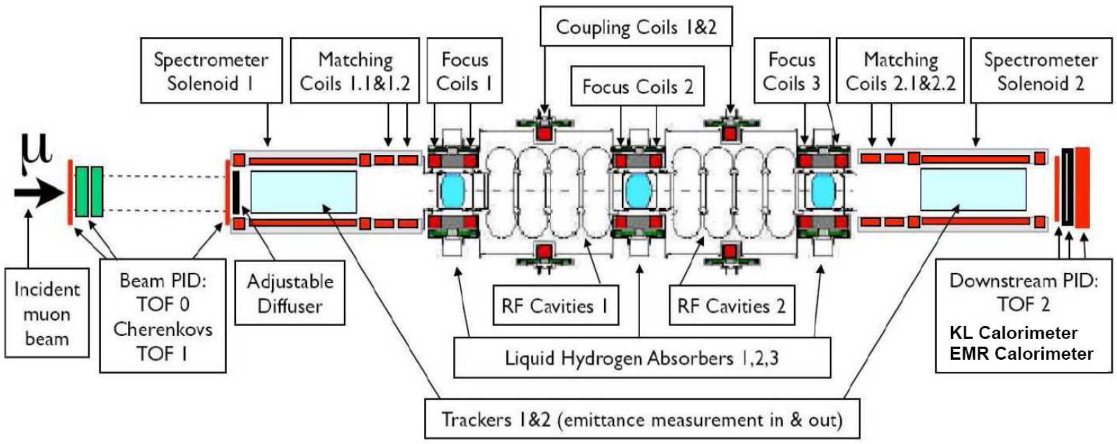

Figure 1: MICE is a staged experiment. The full, final Step VI is shown here. RF cavities and absorbers form the cooling channel, with particle identification and tracking detectors positioned at either end.

The transverse four dimensional (4D) emittance (ϵ_{4D}) is calculated from the fourth root of the determinant of a 4×4 matrix of covariances $\mathbf{V}$:

$$\epsilon_{4D} = \frac{1}{mc}\sqrt[4]{|\mathbf{V}|}, \tag{1}$$

where the elements of $\mathbf{V}$ are $V_{ij} = cov(x_i, x_j)$, with x_i the four transverse phase-space coordinates x, p_x, y, p_y for each muon. In the ideal case of a solenoid field with cylindrical symmetry, the beam Twiss parameters $\alpha_\perp$, $\beta_\perp$ and $\gamma_\perp$ can be extracted from the covariance matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} mc\epsilon_N \frac{\beta_\perp}{p_z} & -mc\epsilon_N\alpha_\perp & 0 & -mc\epsilon_N(\beta_\perp\kappa - \mathcal{L}) \\ -mc\epsilon_N\alpha_\perp & mc\epsilon_N p_z\gamma_\perp & +mc\epsilon_N(\beta_\perp\kappa - \mathcal{L}) & 0 \\ 0 & +mc\epsilon_N(\beta_\perp\kappa - \mathcal{L}) & mc\epsilon_N \frac{\beta_\perp}{p_z} & -mc\epsilon_N\alpha_\perp \\ -mc\epsilon_N(\beta_\perp\kappa - \mathcal{L}) & 0 & -mc\epsilon_N\alpha_\perp & mc\epsilon_N \frac{p_z}{\gamma_\perp} \end{bmatrix} \tag{2}$$

where p_z is the momentum along the beam, κ the radius of curvature of the muon in the solenoidal field, m is the mass of the muon, c the speed of light and ϵ_N the normalised transverse emittance [4]. $\mathcal{L}$ is related to the canonical angular momentum L_{canon} by:

$$\mathcal{L} = \frac{\langle L_{canon}\rangle}{2mc\epsilon_N}.$$

In the cooling channel, p_x and p_y are reduced by energy loss (cooling) and increased by scattering (heating), the latter being the most significant stochastic process. Therefore, if the scattering is small, the samples of muons before and after the cooling channel are highly correlated, and, in the absence of scattering, the statistical error on the relative difference of emittances would be zero[2)].

The fractional change in emittance includes a calculation of the emittance at each tracker (before and after the cooling channel) and is thereby subject to a large degree of correlation between the measurements.

[1)] For brevity, whenever 'emittance' is mentioned we will assume four dimensional transverse normalised emittance ϵ_{4D}.
[2)] Neglecting the Landau fluctuations due to energy loss.

Calculating the statistical error directly is therefore non-trivial. Nevertheless, it is expected that the error should be inversely proportional to the square root of the number of events N. The main aim of this study is to determine this constant of proportionality, to verify empirically the scaling law that governs this error as a function of the beam emittance, by carrying out a large number of simulations.

2 G4MICE Simulations on the Grid

Around 50,000 simulations were completed using the G4MICE [5] software (version 1.9.5) installed on the Grid [6] and were used to calculate the expected emittance measurement at each tracker. These results, shown here in Table 2, were combined to form a distribution of the fractional change in emittance for eight beams, and different numbers of events. The standard deviation of these distributions was found, as was expected, to be inversely proportional to the square root of the number of events N as shown in Figure 2. This constant of proportionality, K, was found for each beam. It will be shown that K is composed of factors arising from applying the standard error propagation formulas to the fractional change in emittance, and a factor due to correlations.

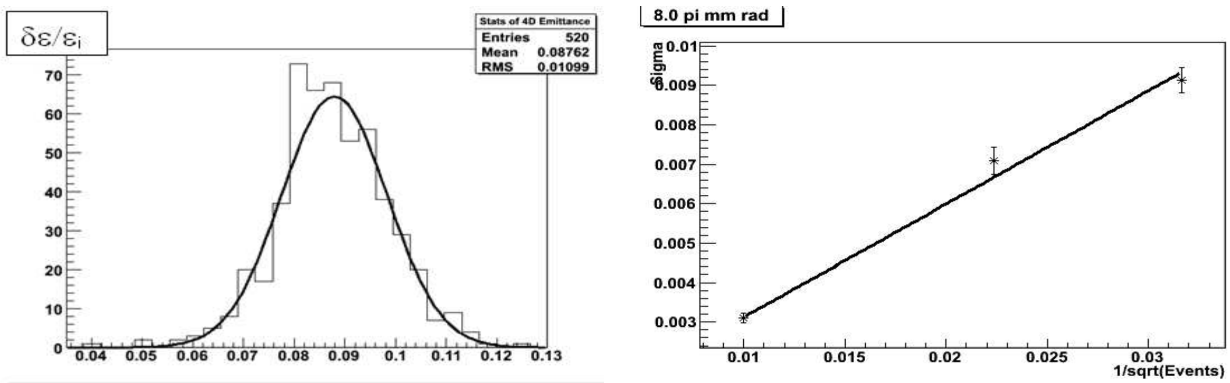

Figure 2: A histogram is constructed using fractional change in emittance results from many simulations (left). Similar histograms are plotted for different numbers of events. These plots are fitted to a Gaussian, yielding standard deviation values which can be plotted against the inverse square root of the number of events (right) to illustrate a proportional relationship.

Let $\delta\epsilon$, ϵ_i and ϵ_o be defined as change in emittance, input emittance and output emittance respectively and the fractional change in emittance f be defined as follows:

$$f = \frac{\delta\epsilon}{\epsilon_i} = \frac{\epsilon_o - \epsilon_i}{\epsilon_i} = \frac{\epsilon_o}{\epsilon_i} - 1 \tag{3}$$

By means of the usual error formulas, we arrive at a standard deviation for f, assuming no correlations:

$$\sigma_f^2 = \left(\frac{\epsilon_o}{\epsilon_i}\right)^2 \left(\left(\frac{\sigma_{\epsilon_o}}{\epsilon_o}\right)^2 + \left(\frac{\sigma_{\epsilon_i}}{\epsilon_i}\right)^2\right), \tag{4}$$

where:

$$\frac{\sigma_{\epsilon_o}}{\epsilon_o} = \frac{\sigma_{\epsilon_i}}{\epsilon_i} = \frac{1}{\sqrt{N}}. \tag{5}$$

Therefore:

$$\sigma_f^2 = \left(\frac{\epsilon_o}{\epsilon_i}\right)^2 \left(\frac{2}{N}\right) = (1+f)^2 \left(\frac{2}{N}\right) \tag{6}$$

$$\sigma_f = K\frac{1}{\sqrt{N}}, \tag{7}$$

where K is defined as follows:

$$K = (1+f)\sqrt{2} \tag{8}$$

Therefore, without yet considering the effect of correlations, K is normally greater than unity, e.g. a fractional drop of 8% gives $f = -0.08$, leading to $K = 1.29$. However, correlations exist between the input and output emittances, since both trackers measure the same sample of muons. Therefore, when we calculate the error derived from the G4MICE simulations σ_{sim}, it includes a correlation factor k_{corr}, that takes into account correlations between the input and output muon samples:

$$\sigma_{sim} = k_{corr}\sigma_f = k_{corr}\,(1+f)\,\sqrt{2}\frac{1}{\sqrt{N}}. \tag{9}$$

We can rearrange for k_{corr}:

$$k_{corr} = \frac{\sigma_{sim}}{\sigma_f} = \sigma_{sim}\sqrt{\frac{N}{2}\frac{\epsilon_i}{\epsilon_o}}. \tag{10}$$

k_{corr} increases in magnitude for very small emittance beams, since they experience heating, an increase in emittance. k_{corr} is always less than or equal to unity for all beams considered and should tend to one for the very small pencil beams, such as the example of the 0.2 π mm rad beam in Table 2 (see also Figure 3). For the small beam, the coefficient K can then have a value greater than unity and therefore require higher statistics for precise measurements than beams with larger emittance, which experience cooling. For high emittance beams the net effect is to pull the value of K below unity, requiring a smaller number of muons to meet our goals than for uncorrelated measurements (see Equation 7 and Figure 3). Assuming a precision requirement of $\sigma_f = 0.01$ we can make a prediction about the number of muons needed for a given beam, as in Table 1. The effect of correlations is to reduce our required number of muons by an order of magnitude for large emittance beams.

3 Conclusions and Future Work

In order to be sure of meeting the goal of measuring 10% change in fractional emittance to an error of 1%, the statistical error should not contribute more than 0.1%. This first study has shown that the number of muons required to meet this target for large emittance beams is of order 10^5, an order of magnitude less than for uncorrelated emittance measurements, due to the correlation factor being less than unity. Since MICE is a single particle experiment, this comes as welcome news ahead of data taking to commence in September 2009.

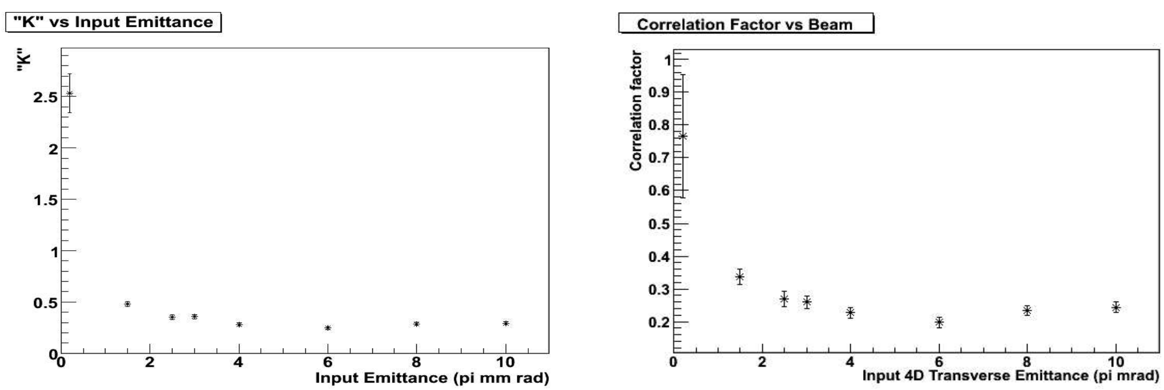

Figure 3: A plot of the proportionality factor K as a function of beam (top). In the case of the pencil beam, K is greater than unity, meaning many more muons are required according to Equation 7. K is partly composed of a correlation factor k_{corr} (bottom) which is less than unity.

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Beam (mm rad)	Without Correlations (10^5 events)	With Correlations (10^5 events)
0.2π	150	64
1.5π	24	2.3
2.5π	20	1.2
3.0π	20	1.3
4.0π	18	0.8
6.0π	17	0.6
8.0π	17	0.8
10.0π	17	0.8

Table 1: The number of muons required for a given beam, in order to achieve a total accuracy of 1% in fractional change in emittance, assuming statistics should not contribute more than 0.1%.

Events	σ	$\delta\sigma$	$\frac{\delta\epsilon}{\epsilon_{in}}$	RMS	Sims
0.2π mm rad					
1000	0.0897	0.00417	1.726	0.1102	450
2000	0.0613	0.00330	1.735	0.0818	261
10000	0.0329	0.00183	1.731	0.0340	242
1.5π mm rad					
1000	0.0168	0.00065	0.084	0.0169	545
2000	0.0106	0.00037	0.0802	0.0110	545
10000	0.0054	0.00018	0.0803	0.0054	545
2.5π mm rad					
1000	0.0117	0.00051	-0.0022	0.0124	421
2000	0.0083	0.00033	-0.0034	0.0092	426
10000	0.0040	0.00025	-0.0050	0.0040	320
3.0π mm rad					
1000	0.0114	0.00048	-0.022	0.0117	513
2000	0.0079	0.00031	-0.025	0.0081	437
10000	0.0036	0.00016	-0.026	0.0036	323
4.0π mm rad					
1000	0.0095	0.00037	-0.046	0.0097	440
2000	0.0066	0.00020	-0.050	0.0068	545
10000	0.0032	0.00015	-0.051	0.0031	340
6.0π mm rad					
1000	0.0073	0.00034	-0.071	0.0079	358
2000	0.0064	0.00023	-0.072	0.0067	500
10000	0.0026	0.00017	-0.072	0.0028	176
8.0π mm rad					
1000	0.0091	0.00031	-0.081	0.0093	549
2000	0.0071	0.00035	-0.083	0.0070	426
10000	0.0031	0.00012	-0.081	0.0032	547
10.0π mm rad					
1000	0.0097	0.00037	-0.081	0.0102	540
2000	0.0069	0.00025	-0.085	0.0068	541
10000	0.0033	0.00016	-0.085	0.0034	359

Table 2: Results for the standard deviation (σ), error in the standard deviation ($\delta\sigma$), fractional change in emittance $\left(\frac{\delta\epsilon}{\epsilon_{in}}\right)$ and RMS for eight beams with different numbers of events. Sims is the number of simulations in the sample. Note that the standard deviation narrows as the number of events increases.

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[6] http://www.gridpp.ac.uk/ Grid Collaboration.